

An easy method for finding the integral of the formula

$$\int \frac{dx}{x} \frac{x^{n+p} - 2x^n \cos \zeta + x^{n-p}}{x^{2n} - 2x^n \cos \theta + 1},$$

with upper limit of integration $x = 1$ or $x = \infty$.

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Nova acta academiae scientarum Petropolitanae 3 (1785), 1788, p. 3–24.

Read on 18 March 1776.

Translation by J. Gélinas, February 2012.

§1. Denoting by S the indefinite integral of this formula, we can ask what will be the value of S , in the case when $x = 1$; first observe that the given formula is much simplified if the numerator and the denominator of the fraction are divided by x^n ; we will then have¹

$$S = \int \frac{dx}{x} \frac{x^p - 2 \cos \zeta + x^{-p}}{x^n - 2 \cos \theta + x^{-n}}. \quad \left\{ \int_0^\infty \frac{\cosh pt + \cos c}{\cosh nt + \cos a} dt \right\}$$

It is immediately obvious that the denominator, $x^n - 2 \cos \theta + x^{-n}$, can always be factored into n factors, each having the form $x^1 - 2 \cos \omega + x^{-1}$, where the angle ω must be chosen so that, if this expression is zero, the denominator is at the same time reduced to zero.

§2. In any case if the factor $x^1 - 2 \cos \omega + x^{-1} = 0$, and thus if $x = \cos \omega + i \sin \omega$, we obtain in general $x^\lambda = \cos \lambda \omega + i \sin \lambda \omega$ and $x^{-\lambda} = \cos \lambda \omega - i \sin \lambda \omega$. Thus the denominator will have this value : $2 \cos n\omega - 2 \cos \theta$, which consequently becomes zero, when we choose for $n\omega$ one of the values :

$$\theta, \theta + 2\pi, \theta + 4\pi, \theta + 6\pi, \theta + 8\pi, \text{etc.}$$

and as the number of these values must be n , all the possible values of the angle ω will be the following :

$$\frac{\theta}{n}, \frac{\theta + 2\pi}{n}, \frac{\theta + 4\pi}{n}, \dots, \frac{\theta + 2(n-1)\pi}{n}.$$

Moreover when $\cos n\omega = \cos \theta$, we will also have $\sin n\omega = \sin \theta$.

§3. Thus, since the denominator has n factors of the form $x^1 - 2 \cos \omega + x^{-1}$, our fraction, whatever its numerator, can be expanded into n simple fractions, whose denominators are the n factors of the denominator. Thus writing for brevity Π to replace the numerator $x^p - 2 \cos \zeta + x^{-p}$, this fraction:

$$\frac{\Pi}{x^n - 2 \cos \theta + x^{-n}}$$

¹ Refer to Appendix D for the formulas inside curly brackets : $a = \pi - \theta, c = \pi - \zeta$ here – translator.

will be decomposed into n simple fractions, each of which having this form :

$$\frac{P}{x^1 - 2 \cos \omega + x^{-1}},$$

and therefore we have established :

$$\frac{\Pi}{x^n - 2 \cos \theta + x^{-n}} = \frac{P}{x^1 - 2 \cos \omega + x^{-1}} + R;$$

where the letter R regroups all the other fractions, and thus we shall have

$$\frac{\Pi (x^1 - 2 \cos \omega + x^{-1})}{x^n - 2 \cos \theta + x^{-n}} = P + R (x^1 - 2 \cos \omega + x^{-1}).$$

If now we make

$$x^1 - 2 \cos \omega + x^{-1} = 0,$$

we find the numerator P ; for it will be

$$P = \frac{\Pi (x^1 - 2 \cos \omega + x^{-1})}{x^n - 2 \cos \theta + x^{-n}},$$

when we make in this equation

$$x = \cos \omega + i \sin \omega.$$

§4. But, then, as we have seen, if we attribute to them this value x , the numerator as well as the denominator of this fraction will be zero, and according to the well known rule², we must write the differential quotients instead of the numerator and the denominator, which gives

$$P = \frac{\Pi (x^1 - x^{-1})}{nx^n - nx^{-n}}.$$

And now, if this chosen value is attributed to x we obtain first for Π this value :

$$\Pi = 2 \cos p\omega - 2 \cos \zeta;$$

and for the other value appearing in the the preceding fraction :

$$\frac{\sin \omega}{n \sin n\omega},$$

thus, since this is a real expression, the required numerator will be

$$P = \frac{2 \sin \omega (\cos p\omega - \cos \zeta)}{n \sin n\omega}.$$

² L'Hôpital's rule in Inst. Calculi Differentialis II – E212, 1755, ch. 15. See: William Dunham, When Euler Met l'Hôpital, Mathematics Magazine, Volume 82, Number 1, February 2009, pp. 16-25 – translator.

We have already seen that

$$\sin n\omega = \sin \theta$$

and this numerator will be

$$P = \frac{2 \sin \omega (\cos p\omega - \cos \zeta)}{n \sin \theta}.$$

§5. So each partial fraction coming from the expansion of the proposed fraction will have this form :

$$\frac{2(\cos p\omega - \cos \zeta)}{n \sin \theta} \frac{\sin \omega}{x^1 - 2 \cos \omega + x^{-1}}$$

from which form if the angle ω is given all the successive values found above, which are

$$\frac{\theta}{n}, \frac{\theta + 2\pi}{n}, \frac{\theta + 4\pi}{n}, \dots, \frac{\theta + 2(n-1)\pi}{n},$$

will appear all the partial fractions, which collected in one sum must produce the proposed form

$$\frac{x^p - 2 \cos \zeta + x^{-p}}{x^n - 2 \cos \theta + x^{-n}}$$

and, which each multiplied by dx/x and integrated, then collected in a single sum, will produce the requested integral S .

§6. So let us consider this fraction :

$$\frac{\sin \omega}{x^1 - 2 \cos \omega + x^{-1}}$$

which multiplied by dx/x gives

$$\frac{\sin \omega dx}{x^2 - 2x \cos \omega + 1},$$

whose integral, taken so that it vanishes at $x = 0$, is certainly

$$\text{Atan} \left(\frac{x \sin \omega}{1 - x \cos \omega} \right).$$

Thus from this partial fraction comes this part of the integral :

$$\frac{2(\cos p\omega - \cos \zeta)}{n \sin \theta} \text{Atan} \left(\frac{x \sin \omega}{1 - x \cos \omega} \right),$$

and now all the n parts of the integral in question can be deduced if all the assigned values are substituted for ω to be collected in the same sum.

§7. But since we are searching for the value of the integral, once substituted $x = 1$, this will produce in this case

$$\text{Atan} \left(\frac{x \sin \omega}{1 - x \cos \omega} \right) = \text{Atan} \left(\frac{\sin \omega}{1 - \cos \omega} \right).$$

But this formula $\sin \omega / (1 - \cos \omega)$ expresses the cotangent of the angle $\omega/2$, and thus the tangent of the angle $(\pi - \omega)/2$, so that this part of the future integral is³

$$\frac{\cos p\omega - \cos \zeta}{n \sin \theta} (\pi - \omega).$$

But here it must be noted in passing that, if we want the integral for the case $x = \infty$, we must take $\text{Atan}(-\sin \omega / \cos \omega)$; thus since $-\sin \omega / \cos \omega$ is the tangent of the angle $\pi - \omega$, while we had before $(\pi - \omega)/2$, it is thus evident that in the case $x = \infty$ the total integral is twice as large as in the case $x = 1$.

§8. Let us assign to ω all its successive values, presented in order here, and we will have

$$\begin{aligned} S = & \frac{\cos p\theta/n - \cos \zeta}{n \sin \theta} \left(\frac{n\pi - \theta}{n} \right) + \frac{\cos p(\theta + 2\pi)/n - \cos \zeta}{n \sin \theta} \left(\frac{(n-2)\pi - \theta}{n} \right) \\ & + \frac{\cos p(\theta + 4\pi)/n - \cos \zeta}{n \sin \theta} \left(\frac{(n-4)\pi - \theta}{n} \right) \\ & + \frac{\cos p(\theta + 6\pi)/n - \cos \zeta}{n \sin \theta} \left(\frac{(n-6)\pi - \theta}{n} \right) + \text{etc.} \end{aligned}$$

where the number of terms must be n . But this expression can be split in two parts, defined as follows :

$$S = \frac{Q}{n \sin \theta} - \frac{R \cos \zeta}{n \sin \theta},$$

so that

$$\begin{aligned} Q = & \frac{n\pi - \theta}{n} \cos \frac{p}{n} \theta + \frac{(n-2)\pi - \theta}{n} \cos \frac{p}{n} (\theta + 2\pi) \\ & + \frac{(n-4)\pi - \theta}{n} \cos \frac{p}{n} (\theta + 4\pi) + \text{etc.} \\ R = & \frac{n\pi - \theta}{n} + \frac{(n-2)\pi - \theta}{n} + \frac{(n-4)\pi - \theta}{n} + \frac{(n-6)\pi - \theta}{n} + \text{etc.} \end{aligned}$$

and so that we must now search for the value of the letters Q and R .

§9. It is immediately clear that the values of this R are in decreasing arithmetic progression with difference $2\pi/n$, and that the sum of n terms will be $\pi - \theta$, so that $R = \pi - \theta$. But the discovery of Q , however, needs a greater development of formulas that we present preceded by general research.

§10. Consider first the progression of cosines, whose angles are increasing in an arithmetic progression and whose number is n :

$$t = \cos(\alpha + 2\beta) + \cos(\alpha + 4\beta) + \cos(\alpha + 6\beta) + \dots + \cos(\alpha + 2n\beta).$$

Now let us multiply both sides by $2 \sin \beta$, and since

$$2 \sin \beta \cos \gamma = \sin(\beta + \gamma) - \sin(\gamma - \beta),$$

³ This formula is not valid, unless $0 < \omega < 2\pi$, and thus follows this condition $0 < \theta < 2\pi$. (Alexander Liapounoff, 1920)

we obtain the following form :

$$2t \sin \beta = -\sin(\alpha + \beta) + \sin(\alpha + 3\beta) + \sin(\alpha + 5\beta) + \dots + \sin(\alpha + (2n + 1)\beta) \\ - \sin(\alpha + 3\beta) - \sin(\alpha + 5\beta) - \dots$$

where all the intermediate terms cancel, so that only the extreme terms remain, thus we will have

$$t = \frac{\sin(\alpha + (2n + 1)\beta) - \sin(\alpha + \beta)}{2 \sin \beta}.$$

§11. Then, moreover, these cosines must be combined with the natural numbers $1, 2, 3, 4, \dots, n$, to get

$$u = 1 \cos(\alpha + 2\beta) + 2 \cos(\alpha + 4\beta) + 3 \cos(\alpha + 6\beta) + \dots + n \cos(\alpha + 2n\beta)$$

this expression being multiplied by $2 \sin \beta$, using the preceding resolution method, we will obtain

$$2u \sin \beta = -\sin(\alpha + \beta) + \sin(\alpha + 3\beta) + 2 \sin(\alpha + 5\beta) + \dots \\ + \sin(\alpha + (2n + 1)\beta) - 2 \sin(\alpha + 3\beta) - 3 \sin(\alpha + 5\beta) - \dots$$

this form being reduced to this one :

$$n \sin(\alpha + (2n + 1)\beta) - 2u \sin \beta = \sin(\alpha + \beta) + \sin(\alpha + 3\beta) + \dots + \sin(\alpha + (2n - 1)\beta)$$

that we denote by v .

§12. Now we multiply also this series by $2 \sin \beta$, and since in general

$$2 \sin \beta \sin \gamma = \cos(\gamma - \beta) - \cos(\gamma + \beta)$$

we will find

$$2v \sin \beta = \cos \alpha - \cos(\alpha + 2\beta) - \cos(\alpha + 4\beta) - \dots - \cos(\alpha + 2n\beta) + \cos(\alpha + 2\beta) + \cos(\alpha + 4\beta) + \dots$$

and thus since all the intermediate terms cancel each other, we obtain

$$v = \frac{\cos \alpha - \cos(\alpha + 2n\beta)}{2 \sin \beta};$$

and since

$$u = \frac{n \sin(\alpha + (2n + 1)\beta) - v}{2 \sin \beta}$$

we have thus obtained

$$u = \frac{n \sin(\alpha + (2n + 1)\beta)}{2 \sin \beta} - \frac{\cos \alpha - \cos(\alpha + 2n\beta)}{4 \sin^2 \beta}.$$

§13. Let us combine the two sums using the general method, and we establish

$$V = (a + b) \cos(\alpha + 2\beta) + (a + 2b) \cos(\alpha + 4\beta) + (a + 3b) \cos(\alpha + 6\beta) + \dots + (a + nb) \cos(\alpha + 2n\beta),$$

and it is evident that $V = at + bu$, and that by replacing t and u by their values we will obtain

$$V = \frac{a \sin(\alpha + (2n + 1)\beta) - a \sin(\alpha + \beta)}{2 \sin \beta} + \frac{bn \sin(\alpha + (2n + 1)\beta)}{2 \sin \beta} - \frac{b \cos \alpha - b \cos(\alpha + 2n\beta)}{4 \sin^2 \beta}$$

§14. Now it is very clear, that the progression that we have denoted by Q above, is contained in the general form V , that the same number of terms occurs on both sides, and that the coefficients of the series of cosines Q constitute an arithmetic progression. In this fashion let us make first for the coefficients

$$a + b = \frac{n\pi - \theta}{n}$$

and

$$a + 2b = \frac{(n - 2)\pi - \theta}{n}$$

thus we deduce

$$b = -\frac{2\pi}{n}$$

and

$$a = \frac{(n + 2)\pi - \theta}{n}$$

Now let us make the angles be proportional to each other

$$\alpha + 2\beta = \frac{p}{n}\theta$$

and

$$\alpha + 4\beta = \frac{p}{n}(\theta + 2\pi)$$

and we deduce that $\beta = \pi p/n$, and moreover

$$\alpha = \frac{p}{n}(\theta - 2\pi) = -\frac{p}{n}(2\pi - \theta)$$

in this way we make $V = Q$. But the angles occurring in this expression V will be : first

$$\alpha + (2n + 1)\beta = -\frac{p}{n}(\pi - \theta)$$

where since p is integer number, the last part $2\pi p$ is omitted to express the total circumference, from which we get

$$\sin(\alpha + (2n + 1)\beta) = -\sin \frac{p}{n}(\pi - \theta).$$

Then we have the angle

$$\alpha + \beta = -\frac{p}{n}(\pi - \theta)$$

whose sinus is

$$\sin(\alpha + \beta) = -\sin \frac{p}{n}(\pi - \theta).$$

Finally we have

$$\alpha + 2n\beta = -\frac{p}{n}(2\pi - \theta),$$

and

$$\cos(\alpha + 2n\beta) = \cos \frac{p}{n}(2\pi - \theta).$$

The substitution of these values produces

$$\begin{aligned} Q = V = & \frac{-\frac{(n+2)\pi-\theta}{n} \sin p(\pi - \theta)/n + \frac{(n+2)\pi-\theta}{n} \sin p(\pi - \theta)/n}{\sin p\pi/n} \\ & + \frac{2\pi \sin p(\pi - \theta)/n}{2 \sin p\pi/n} + \frac{\frac{2\pi}{n} \cos p(2\pi - \theta)/n - \frac{2\pi}{n} \cos p(2\pi - \theta)/n}{4 \sin^2 p\pi/n}, \end{aligned}$$

this expression clearly being reduced to this :

$$Q = V = \frac{\pi \sin p(\pi - \theta)/n}{\sin p\pi/n}.$$

§15. Having found the value of the letters Q and R , the value of the integral considered⁴ for the case $x = 1$ will be

$$S = \frac{\pi \sin p(\pi - \theta)/n}{n \sin \theta \sin p\pi/n} - \frac{(\pi - \theta) \cos \zeta}{n \sin \theta}. \quad \left\{ \frac{1}{n} \frac{\pi \sin ab}{\sin a \sin \pi b} + \frac{\cos c}{n} \frac{a}{\sin a} \right\}$$

But if the requested integral is taken from $x = 0$ to $x = \infty$, its value will be twice as large.

§16. These general cases having been treated, let us consider now those often met when $\zeta = 90^\circ$ and $\theta = 90^\circ$, so that the proposed integral formula is

$$\int \frac{dx}{x} \frac{x^p + x^{-p}}{x^n + x^{-n}}, \quad \left\{ \int_0^\infty \frac{\cosh pt}{\cosh nt} dt \right\}$$

and thus we will have for the case $x = 1$ the value

$$S = \frac{\pi \sin p\pi/2n}{n \sin p\pi/n},$$

which since

$$\sin \frac{p\pi}{n} = 2 \sin \frac{p\pi}{2n} \cos \frac{p\pi}{2n},$$

⁴ Refer to Appendix D for the formulas inside curly brackets : $b = p/n$ here – translator.

gives us a noteworthy formula

$$\frac{\pi}{2n \cos p\pi/2n} = \frac{\pi}{2n} \sec \frac{p\pi}{2n}. \quad \left\{ \frac{1}{n} \frac{\pi}{2} \sec \frac{\pi b}{2} \right\}.$$

But if we only take $\zeta = 90^\circ$ so that the formula to be integrated is

$$\int \frac{dx}{x} \frac{x^p + x^{-p}}{x^n - 2 \cos \theta + x^{-n}}, \quad \left\{ \int_0^\infty \frac{\cosh pt}{\cosh nt + \cos a} dt \right\}$$

its value taken from $x = 0$ up to $x = 1$ will be

$$\frac{\pi \sin p(\pi - \theta)/n}{n \sin \theta \sin p\pi/n}, \quad \left\{ \frac{1}{n} \frac{\pi \sin ab}{\sin a \sin \pi b} \right\}$$

this expression being reduced to this :

$$\frac{\pi}{n \sin \theta} \left(\cos \frac{p\theta}{n} - \sin \frac{p\theta}{n} \cot \frac{p\pi}{n} \right).$$

Observations on usual integration.

§I. First, let us note that the middle term in the numerator does not inhibit the integration since if it was alone the integration would not pose any problems; indeed

$$\int \frac{dx}{x} \frac{1}{x^n - 2 \cos \theta + x^{-n}} \quad \left\{ \frac{1}{2} \int_0^\infty \frac{dt}{\cosh nt + \cos a} \right\}$$

is reduced to this form :

$$\int \frac{x^{n-1} dx}{x^{2n} - 2x^n \cos \theta + 1}, \quad \left\{ \int_0^\infty \frac{e^{-nt} dt}{e^{-2nt} + 2e^{-nt} \cos a + 1} \right\}$$

which after the substitution $x^n = y$ becomes this one

$$\frac{1}{n} \int \frac{dy}{y^2 - 2y \cos \theta + 1}, \quad \left\{ \frac{1}{n} \int_0^1 \frac{dy}{y^2 + 2y \cos a + 1} \right\}$$

whose integral is

$$\frac{1}{n \sin \theta} \text{Atan} \left(\frac{y \sin \theta}{1 - y \cos \theta} \right), \quad \left\{ \frac{1}{n \sin a} \text{Atan} \left(\frac{y \sin a}{1 + y \cos a} \right) \Big|_0^1 \right\}$$

whose value, for the case $x = 1$, will be

$$\frac{1}{n \sin \theta} \text{Atan} \left(\frac{\sin \theta}{1 - \cos \theta} \right) = \frac{\pi - \theta}{2n \sin \theta} \quad \left\{ \frac{1}{2n} \frac{a}{\sin a} \right\}$$

which when multiplied by $-2 \cos \zeta$ gives a result of the form found above, thus the resulting term will be kept in the computation, and we are led to look at this integral :

$$\int \frac{dx}{x} \frac{x^p + x^{-p}}{x^n - 2 \cos \theta + x^{-n}} \quad \left\{ \int_0^\infty \frac{\cosh pt}{\cosh nt + \cos a} dt \right\}$$

whose value, for the case $x = 1$, we have found to be

$$= \frac{\pi \sin p(\pi - \theta)/n}{n \sin \theta \sin p\pi/n}, \quad \left\{ \frac{1}{n} \frac{\pi \sin ab}{\sin a \sin \pi b} \right\}$$

that for brevity we denote by the letter P , so that

$$P = \frac{\pi \sin p(\pi - \theta)/n}{n \sin \theta \sin p\pi/n}.$$

And, effectively, we also found that in the case $x = \infty$ the value of this formula is $2P$.

§II. Secondly, it must be noted that the exponent p must necessarily be smaller than the exponent n since otherwise the fraction would be improper and the variable x would have more degrees in the numerator than in the denominator. When this happens, in addition to the terms integrated from the partial fractions

there must be added others, which were not taken into account in our solution, thus this is why such cases must totally excluded. But in all cases, it will be easy to add these terms to the parts provided by our method.

§III. In this solution, that we have given, it is clear that the exponent p must necessarily be an integer since otherwise the operations done could not take place; and it will be seen as marvelous that the conclusions found can stand, even when this exponent p will be any fraction, although smaller than n since these cases can be reduced to the case of integer exponents. To show this let $p = q/\lambda$, so that our formula, substituting $x = z^\lambda$, will be reduced to this form :

$$\lambda \int \frac{dz}{z} \frac{z^q + z^{-q}}{z^{\lambda n} - 2 \cos \theta + z^{-\lambda n}}$$

where since all the exponents are integers, and the limits of integration, which were $x = 0$, $x = 1$ and $x = \infty$ become then $z = 0$, $z = 1$ and $z = \infty$, the value of the integral for $z = 1$ will be

$$\frac{\lambda \pi \sin q(\pi - \theta)/\lambda n}{\lambda n \sin \theta \sin q\pi/\lambda n},$$

which, when q/λ is replaced by its value p , becomes this

$$\frac{\pi \sin p(\pi - \theta)/n}{n \sin \theta \sin p\pi/n},$$

an expression that is in total accordance with the preceding. And thus it is also agreed that when irrational values are attributed to the exponent p , provided they do not exceed the exponent n , this will still happen.

§IV. Here must be asked a very important question, is it licit or not to give imaginary values to the exponent p ? But it will be seen that we can answer affirmatively, since the imaginaries are certainly not greater than n ; and we conclude, on the condition that the value of this p be taken as imaginary, so that this differential formula

remains real, that our conclusions also remain valid⁵. This happens, if we set $p = iq$; since then, as we have in general

$$e^{i\phi} + e^{-i\phi} = 2 \cos \phi,$$

which in our case is $\phi = q \log x$, the integral formula itself will be

$$\int \frac{dx}{x} \frac{2 \cos(q \log x)}{x^n - 2 \cos \theta + x^{-n}}.$$

Now we see what our integral formula becomes in the case $x = 1$, and for sinus of imaginary angles which are also imaginary, since

$$e^{i\phi} - e^{-i\phi} = \frac{2}{i} \sin \phi,$$

instead of ϕ let us write $i\psi$, so that

$$\sin i\psi = \frac{i}{2}(e^{-\psi} - e^{+\psi}),$$

⁵ Although this is not sufficiently justified, it is still in fact permitted to put here $p = iq$, since the integral, if the absolute value of p is kept small enough, can be developed in terms of this p as a series of successive integer powers. (Alexander Liapounoff, 1920)

and in the integral form we will have

$$\frac{p}{n} = i \frac{q}{n},$$

similarly in place of ψ let us write $\frac{q}{n}(\pi - \theta)$ in the numerator, and $\frac{q\pi}{n}$ in the denominator, from which the value of the integral taken from $x = 0$ up to $x = 1$ will be

$$\frac{\pi}{n \sin \theta} \frac{e^{-\frac{q}{n}(\pi - \theta)} - e^{+\frac{q}{n}(\pi - \theta)}}{e^{-\frac{q\pi}{n}} - e^{+\frac{q\pi}{n}}}.$$

Thus let us formulate the following noteworthy theorem :

If the integral formula :

$$\int \frac{dx}{x} \frac{\cos(q \log x)}{x^n - 2 \cos \theta + x^{-n}} \quad \left\{ \frac{1}{2} \int_0^\infty \frac{\cos qt}{\cosh nt + \cos a} dt \right\}$$

is taken from the limit $x = 0$ to $x = 1$, its value will always be

$$\frac{\pi}{2n \sin \theta} \frac{e^{-\frac{q}{n}(\pi - \theta)} - e^{+\frac{q}{n}(\pi - \theta)}}{e^{-\frac{q\pi}{n}} - e^{+\frac{q\pi}{n}}}. \quad \left\{ \frac{1}{2n \sin a \sinh(\pi q/n)} \right\}$$

**But if the integral is taken from $x = 0$ up to $x = \infty$,
its value will be twice more.**

Of course, this theorem deserves greater attention, so that nothing would be missing from the demonstration of its validity.

§V. But let us revert to the integral form presented first, whose numerator has two terms x^p and x^{-p} , and whose integral sum for $x = 1$ was found to be $= P$, for the case $x = \infty$ twice more $= 2P$, it is particularly noteworthy here, that for the limit $x = \infty$ these two parts of the numerator produce a value $= P$. We always have in fact, taking the integral from $x = 0$ to $x = \infty$,

$$\int \frac{dx}{x} \frac{x^{+p}}{x^n - 2 \cos \theta + x^{-n}} = \int \frac{dx}{x} \frac{x^{-p}}{x^n - 2 \cos \theta + x^{-n}} = P.$$

To show it, let us put in the lat formula $x = 1/z$, and this form appears :

$$- \int \frac{dz}{z} \frac{z^{+p}}{z^{-n} - 2 \cos \theta + z^{+n}},$$

which as it is totally similar to the previous form, the sign $-$ excepted, has a value from the limit $z = 0$ to $z = \infty$, when negated, equal to the first formula. But as $z = 1/x$, its limits of integration being from $x = \infty$ to $x = 0$, which when interchanged, change the sign of the integral, it is equal to the previous formula; that is why as the two formulas conjugated have sum $2P$, each one has a value equal to P , and we deduce the following theorem worthy to be also known :

The value of this integral formula :

$$\int \frac{dx}{x} \frac{x^{\pm p}}{x^n - 2 \cos \theta + x^{-n}}, \quad \left\{ \frac{1}{2} \int_{-\infty}^\infty \frac{e^{\pm pt} dt}{\cosh nt + \cos a} \right\}$$

taken from the bound $x = 0$ up to $x = \infty$ is always

$$= P = \frac{\pi \sin p(\pi - \theta)/n}{n \sin \theta \sin p\pi/n} \cdot \left\{ \frac{1}{n} \frac{\pi \sin ab}{\sin a \sin \pi b} \right\}$$

Clearly however this equality cannot hold for the case $x = 1$.

§VI. Whereas in our differential formula there appears only the term $2 \cos \theta$, whose value remains identical, when we take $\theta \pm 2k\pi$ to replace θ , it must seem quite surprising that a very different value for the integral is produced, which is

$$= \frac{\pi \sin (\pi - \theta \pm 2k\pi)p/n}{n \sin \theta \sin \pi p/n},$$

and it is worth asking which of these values is exact, to which nothing else can be answered, except the truth that all are to be considered equally valid⁶, and which must seem less strange, that all these integral formulas are multiform functions, and thus infiniteform, what this simple example : $\int \frac{dx}{1+x^2}$ can help to understand. As in fact this integral produces the circular arc whose tangent is x , but as innumerable arcs can be given whose tangent is x , it is necessary that all be contained in the integrated form. But in our explicit value for P , in place of π we can write also write $\pi + 2k\pi$, nevertheless its value can remain valid. However in this kind of integrals the minimal values are usually wanted, so that all difficulties are taken out of the formula.

§VII. In the analysis presented above, we have supposed the factors of the denominator unequal to each other, which happens, unless $\cos \theta = \pm 1$, in which cases the denominator involves squares : it is then

$$= x^{-n}(x^n \pm 1)^2;$$

from which follows that all factors $x^n \pm 1$ must appear twice. This problem also implies for our formula P

that the case $\theta = 0$ signals an infinite value. But putting $\theta = \pi$, a singular phenomenon happens, when in the formula found for P both the numerator and the denominator vanish, so that we must determine the value of their quotient. Let us put to that effect $\theta = \pi - \omega$, for an infinitely small ω , and we will have

$$\sin \theta = \sin \omega = \omega;$$

and since $\pi - \theta = \omega$, we will have in the numerator

$$\sin \frac{p\omega}{n} = \frac{p\omega}{n},$$

and this value of P results

$$\frac{\pi p}{nn \sin \frac{\pi p}{n}},$$

which as it is fully determined, there can remain no doubt that it is valid, and that emerges the following theorem amongst the most memorable :

Theorem. Consider the differential formula

$$\frac{dx}{x} \frac{x^p + x^{-p}}{x^n + 2 + x^{-n}} = \frac{x^{n-1} dx (x^p + x^{-p})}{(x^n + 1)^2}, \quad \left\{ \int_0^\infty \frac{\cosh(2pt)}{\cosh^2(nt)} dt \right\}$$

⁶ the proposed formula is valid only if $0 < \theta < 2\pi$. (Alexander Liapounoff, 1920)

whose integral is taken from $x = 0$ to $x = 1$, its value will always be

$$\frac{\pi p/n}{n \sin \pi p/n}; \quad \left\{ \frac{1}{n} \frac{\pi b}{\sin \pi b} \right\}$$

but if taken up to $x = \infty$, its value will be twice greater,

$$\frac{2\pi p/n}{n \sin \pi p/n}.$$

Direct demonstration of this theorem.

This integral formula can be decomposed thus :

$$\int \frac{dx}{x} \frac{x^{n+p} + x^{n-p}}{(1+x^n)^2} = \frac{Q}{1+x^n} + \int \frac{dx}{x} \frac{R}{1+x^n}.$$

Taking thus this differential multiplied by x/dx , and denoting $dQ = Q'dx$, this equation will appear :

$$\frac{x^{n+p} + x^{n-p}}{(1+x^n)^2} = \frac{Q'x}{1+x^n} - \frac{nQx^n}{(1+x^n)^2} + \frac{R}{1+x^n},$$

which multiplied by $1+x^n$ can be represented in this fashion :

$$\frac{x^{n+p} + x^{n-p} + nQx^n}{1+x^n} = Q'x + R,$$

where now Q must be taken so that this fractional expression becomes entire. But this is easy to do this by putting $nQ = -x^p + x^{n-p}$, since then this fraction must be

$$\frac{x^{n-p} + x^{2n-p}}{1+x^n} = x^{n-p},$$

so that we now have $x^{n-p} = Q'x + R$. Since

$$Q = \frac{x^{n-p} - x^p}{n},$$

we will have

$$Q'x = \frac{(n-p)x^{n-p} - px^p}{n},$$

thus we get $R = p(x^{n-p} + x^p)/n$, and so the proposed integral formula is reduced to this form :

$$\frac{x^{n-p} - x^p}{n(1+x^n)} + \frac{p}{n} \int \frac{dx}{x} \frac{x^{n-p} + x^p}{1+x^n}$$

this integral being taken so as to vanish for $x = 0$. Now we put $x = 1$, so that the preceding integrated part vanishes, and the value of the integral formula, from what is known since a long time⁷, becomes

$$\frac{p}{n} \frac{\pi}{n \sin p\pi/n} = \frac{\pi p}{nn \sin p\pi/n},$$

which thus is in perfect accord with what has been found before.

§VIII. If we attribute in this last formula imaginary values to the exponent p , putting $p = iq$, and since, as seen earlier, this makes $x^p + x^{-p} = 2 \cos(q \log x)$, the proposed integral formula will be

$$= 2 \int \frac{dx}{x} \frac{x^n \cos(q \log x)}{(1 + x^n)^2}.$$

For its value we have already seen that

$$\sin \frac{i\pi q}{n} = \frac{e^{-\pi q/n} - e^{+\pi q/n}}{2i}$$

and thus the value of our formula, taken from $x = 0$ to $x = 1$, will be

$$\frac{2\pi q}{nn(e^{\pi q/n} - e^{-\pi q/n})}$$

and we deduce the following theorem worthy of all attention.

Theorem.

The value of the integral formula :

$$\int \frac{dx}{x} \frac{x^n \cos(q \log x)}{(1 + x^n)^2}, \quad \left\{ \frac{1}{2} \int_0^\infty \frac{\cos(2qt)}{\cosh^2(nt)} dt \right\}$$

taken from $x = 0$ up to $x = 1$, is always equal to this formula :

$$\frac{\pi q/n}{n(e^{\pi q/n} - e^{-\pi q/n})}. \quad \left\{ \frac{1}{2n} \frac{\pi q/n}{\sinh(\pi q/n)} \right\}$$

The demonstration of this theorem could seem difficult to draw from principles already known.

§IX. And it is also clear that the method that we have used to integrate our formula cannot function without the middle term being smaller, and that is why we have given it the form $2 \cos \theta$. For this reason the important question must be asked : whether our conclusions would still hold if this middle term would be larger, if the angle θ was imaginary, or not ? But without doubt even this case can subsist, but the final accord on the validity of our formula is yet to come. Before anything else, however, it must be pointed out that negative values of this middle term $2 \cos \theta$ cannot be so attributed that would make the denominator vanish, when the value of our variable x is increased from 0 to 1. For this reason let us set the angle $\theta = \pi - \eta$, and the value of our integral will be

$$\int_{x=0}^{x=1} \frac{dx}{x} \frac{x^p + x^{-p}}{x^n + 2 \cos \eta + x^{-n}} = \frac{\pi \sin p\eta/n}{n \sin \eta \sin p\pi/n}.$$

⁷ Inst. Calculi Integralis IV – E660, Supp. III, §70, p. 123; or §16 with $p - n/2, n/2$ – translator.

Thus in that formula let us make the angle η imaginary, defining $\eta = \phi i$, so that by what we found above, $2 \cos \phi i = e^\phi + e^{-\phi}$, and so that our denominator becomes

$$x^n + e^\phi + e^{-\phi} + x^{-n} = \frac{1}{x^n} (x^n + e^\phi)(x^n + e^{-\phi})$$

which thus can be resolved immediately in two real factors of the form $x + k$; then in fact we must make

$$\sin \eta = \sin \phi i = \frac{e^{-\phi} - e^{+\phi}}{2i},$$

similarly we will have

$$\sin \frac{p\eta}{n} = \sin \frac{p\phi i}{n} = \frac{e^{-p\phi/n} - e^{+p\phi/n}}{2i},$$

and our integral formula becomes real

$$= \frac{\pi(e^{-p\phi/n} - e^{+p\phi/n})}{n(e^{-\phi} - e^{+\phi}) \sin p\pi/n}.$$

We define here for brevity $e^\phi = f$, so that $e^{-\phi} = 1/f$, and our integral formula gets the following form :

$$\int_{x=0}^{x=1} \frac{dx}{x} \frac{x^p + x^{-p}}{x^n + f + 1/f + x^{-n}} = \frac{\pi(f^{p/n} - f^{-p/n})}{n(f - f^{-1}) \sin \pi p/n},$$

a theorem that can be seen worthy of all attention; and it is understood that the value of this integral, taken up to $x = \infty$, would be twice as large.

§X. Now if in this form we give to the exponent p imaginary values, similarly there can be no doubt that our conclusion remains valid. Let us put in this case $p = qi$, so that as before $x^p + x^{-p} = 2 \cos(q \log x)$; then we will have

$$\sin \frac{p\pi}{n} = \frac{e^{-q\pi/n} - e^{q\pi/n}}{2i}$$

but for the numerator of the integral we will have

$$f^{p/n} - f^{-p/n} = 2i \sin \frac{q \log f}{n}.$$

From these values we find what follows

Theorem.

The value of this integral formula :

$$\int \frac{dx}{x} \frac{\cos(q \log x)}{x^n + f + 1/f + x^{-n}},$$

taken from $x = 0$ up to $x = 1$, is always equal to the formula

$$\frac{2\pi \sin q(\log f)/n}{n(f - f^{-1})(e^{q\pi/n} - e^{-q\pi/n})}.$$

§XI. Then, as we observed long ago⁸, all such integrals can be easily expressed by infinite series. Since this fraction :

$$\frac{x^p}{x^n - 2 \cos \theta + x^{-n}} = \frac{x^{n+p}}{x^{2n} - 2x^n \cos \theta + 1}$$

can be developed into this series :

$$\frac{1}{\sin \theta} (x^{n+p} \sin \theta + x^{2n+p} \sin 2\theta + x^{3n+p} \sin 3\theta + \text{etc.})$$

thus the integral :

$$\int \frac{dx}{x} \frac{x^p}{x^n - 2 \cos \theta + x^{-n}}, \quad \left\{ \frac{1}{2} \int_0^\infty \frac{e^{pt}}{\cosh nt - \cos \theta} dt \right\}$$

taken from $x = 0$ up to $x = 1$, is equal to this infinite series :

$$\frac{1}{\sin \theta} \left(\frac{\sin \theta}{n+p} + \frac{\sin 2\theta}{2n+p} + \frac{\sin 3\theta}{3n+p} + \frac{\sin 4\theta}{4n+p} + \text{etc.} \right). \quad \left\{ \frac{1}{n \sin \theta} \sum_{k=1}^\infty \frac{\sin k\theta}{k+b} \right\}$$

And thus, if we make p negative, then our principal formula

$$\int \frac{dx}{x} \frac{x^p + x^{-p}}{x^n - 2 \cos \theta + x^{-n}}, \quad \left\{ \int_0^\infty \frac{\cosh pt}{\cosh nt - \cos \theta} dt \right\}$$

taken from $x = 0$ to $x = 1$, is always equal to this double infinite series :

$$\begin{aligned} \frac{1}{\sin \theta} \left(\frac{\sin \theta}{n+p} + \frac{\sin 2\theta}{2n+p} + \frac{\sin 3\theta}{3n+p} + \frac{\sin 4\theta}{4n+p} + \text{etc.} \right. \\ \left. + \frac{\sin \theta}{n-p} + \frac{\sin 2\theta}{2n-p} + \frac{\sin 3\theta}{3n-p} + \frac{\sin 4\theta}{4n-p} + \text{etc.} \right), \end{aligned}$$

which after grouping two by two similar terms contracts into this series :

$$\frac{2n}{\sin \theta} \left(\frac{\sin \theta}{nn - pp} + \frac{2 \sin 2\theta}{4nn - pp} + \frac{3 \sin 3\theta}{9nn - pp} + \frac{4 \sin 4\theta}{16nn - pp} + \text{etc.} \right). \quad \left\{ \frac{2}{n \sin \theta} \sum_{k=1}^\infty \frac{k \sin k\theta}{k^2 - b^2} \right\}$$

§XIIa. Now it is clear that by putting $p = qi$, this series will be obtained :

$$\frac{2n}{\sin \theta} \left(\frac{\sin \theta}{nn + qq} + \frac{2 \sin 2\theta}{4nn + qq} + \frac{3 \sin 3\theta}{9nn + qq} + \frac{4 \sin 4\theta}{16nn + qq} + \text{etc.} \right) \quad \left\{ \frac{2}{n \sin \theta} \sum_{k=1}^\infty \frac{k \sin k\theta}{k^2 + (q/n)^2} \right\}$$

⁸ Leonhard Euler, Inst. Calculi Integralis IV (1845) – Eneström 660, Supplementum V, §188, p. 373.

which thus expresses the value of this integral formula :

$$\int \frac{dx}{x} \frac{2 \cos(q \log x)}{x^n - 2 \cos \theta + x^{-n}}, \quad \left\{ \int_0^\infty \frac{\cos qt}{\cosh nt - \cos \theta} dt \right\}$$

taken from $x = 0$ to $x = 1$, so that the sum of this series can be expressed in finite terms

$$\frac{\pi}{n \sin \theta} \left(\frac{e^{-q(\pi-\theta)/n} - e^{+q(\pi-\theta)/n}}{e^{-q\pi/n} - e^{+q\pi/n}} \right), \quad \left\{ \frac{\pi \sinh(\pi - \theta)q/n}{n \sin(\pi - \theta) \sinh \pi q/n} \right\}$$

And it is easily understood that the angle θ can take imaginary values. We have seen that after putting $\theta = \phi i$ we would have

$$\sin \theta = \frac{e^{-\phi} - e^{+\phi}}{2i},$$

and thus in general

$$\sin \lambda \theta = \frac{e^{-\lambda \phi} - e^{+\lambda \phi}}{2i}.$$

And so if we put $e^\phi = f$, we will have

$$\frac{\sin \lambda \theta}{\sin \theta} = \frac{f^\lambda - f^{-\lambda}}{f - 1/f},$$

and that series has a concise enough form.

§XIIb. Finally the operations, which we have used for our integral formulas, fail unless the exponent n is an integer number. However, the value of the integrals that we have found for the cases $x = 1$ or $x = \infty$, remains valid not only when n takes on arbitrary fractional values but also imaginary values, which can be shown easily in the first case. Thus let $n = m/\lambda$, and put $x = y^\lambda$, so that

$$\frac{dx}{x} = \lambda \frac{dy}{y},$$

and there appears this integral form containing integer exponents :

$$\lambda \int \frac{dy}{y} \frac{y^{\lambda p} + y^{-\lambda p}}{y^m - 2 \cos \theta + y^{-m}}$$

whose value in the case $x = 1$ must be the second formula found above

$$\frac{\lambda \pi}{m} \frac{\sin \lambda p(\pi - \theta)/m}{\sin \theta \sin \lambda p \pi / m},$$

which, when m is replaced by its value λn , clearly gives our formula found above :

$$\frac{\pi}{n} \frac{\sin p(\pi - \theta)/n}{\sin \theta \sin p \pi / n}.$$

In the other case there is no doubt that their validity remains, when n takes imaginary values⁹. Thus, let us put $n = mi$; and the integral formula will be reduced to this form :

$$\int \frac{dx}{x} \frac{x^p + x^{-p}}{2 \cos(m \log x) - 2 \cos \theta},$$

⁹ It is manifest that this conclusion is false, whenever we take for n an imaginary number of the form im . (Alexander Liapounoff, 1920)

whose value for the case $x = 1$ will certainly be

$$\frac{\pi}{mi} \frac{e^{p(\pi-\theta)/m} - e^{-p(\pi-\theta)/m}}{\sin \theta (e^{p\pi/m} - e^{-p\pi/m})}$$

where it will be considered surprising that this value be always imaginary, although this differential form, for the variable x , from the limit $x = 0$ up to the limit $x = 1$, stays real, which merits to be seen as a great paradox. Meanwhile however cases are not lacking where the value of an integral of real differentials becomes clearly imaginary, which in this simple formula

$$\int \frac{dx}{x \cos(m \log x)}$$

is sufficiently demonstrated, although of course, when x increases from 0 to 1, it stays constantly real. To integrate this formula, let $\log x = -z$, where it must be noted that, when x progresses from 0 up to 1, then the quantity z decreases from ∞ to 0. Now our integral formula will be

$$\int \frac{-dz}{\cos mz},$$

as we see that

$$\int \frac{d\phi}{\sin \phi} = \log \tan \frac{\phi}{2},$$

let us take $\phi = 90^\circ - mz$, we will have $d\phi = -mdz$, and

$$\int \frac{-mdz}{\cos mz} = + \log \tan(45^\circ - \frac{mz}{2}),$$

this integral vanishing for the limit $z = 0$, but the quantity z increasing from this limit to infinity, the tangent of the angle being negative and the logarithm being imaginary, so we do not need to ask the question of how integrals of real differential formulas become imaginary in certain cases.

§XIII. In this manner, we are assured that our proposed differential formula

$$\int \frac{dx}{x} \frac{x^p + x^{-p}}{x^n - 2 \cos \theta + x^{-n}}, \quad \left\{ = \frac{\pi \sin(\pi - \theta)p/n}{n \sin(\pi - \theta) \sin \pi p/n} \right\}$$

for the integral taken from $x = 0$ to $x = 1$ is always valid, for any values attributed to the three letters n, p and θ , either integers, fractions, or even imaginary numbers. At the same time, however, in the cases which have been indicated above, are excluded the values which render aberrant the value of the integral each time that the exponent p is larger than the exponent n and we must also exclude all cases where the real part of $p - n$ is positive. But that excepted, we have found diverse formulas that can be considered to be worthy of the greatest attention, without neglecting the increase in knowledge for the science of analysis.

Appendix A – Modifications for the translation from the Latin

1. i replaces $\sqrt{-1}$ (T_EX macro I).
2. k replaces i when used as an index.
3. \log replaces l for the natural logarithm.
4. dx replaces ∂x as differential.
5. $/4 \sin^2(\cdot)$ replaces $/4 \sin(\cdot)^2$ (§12, §13, §14).
6. $-b \cos(\alpha + 2n\beta)$ replaces $+b \cos(\alpha + 2n\beta)$ (last equation, §13).
7. $Q = V = (-. + .)/.$ replaces $Q = V = -(. + .)/.$ (second to last equation, §14).
8. $x^n + 2 + x^{-n}$ replaces $x^n - 1 + x^{-n}$ (theorem, §VII).
9. $dQ = Q'dx$ replaces $dQ' = Qdx$ (demonstration, §VII).
10. $x^{iq} + x^{-iq}$ replaces $x^p + x^{-p}$ (theorem, §VIII).
11. $x^n - 2 \cos \theta + x^{-n}$ replaces $x^{2n} - 2x^n \cos \theta + 1$ (§XI).
12. $\sin \eta$ replaces $\sin \theta$ (§XI).
13. $2 \cos(q \log x)$ replaces $\cos(q \log x)$ (§XIIa).
14. π/mi replaces p/mi (§XIIb).

The corrections 5–13 have also been made by Alexander Liapounoff for the re-edition in 1920 of the article in volume XVIII of the series 1 of the Opera Omnia. The article in Latin is prefaced there by the summary of 1788 in French.

Appendix B – Summary and proofs by Siméon Denis Poisson [a]

§22. The simplest formulas included in the preceding ones are those obtained in putting $\theta = 0$ in the equations (3) and (7). If at the same time $2m$ is substituted instead of m in the first one, and that the reductions are done, one obtains¹⁰

$$\int \frac{e^{2mt} - e^{-2mt}}{e^{\pi t} - e^{-\pi t}} dt = \frac{1}{2} \tan m;$$

$$\int \frac{e^{2mt} + e^{-2mt}}{e^{\pi t} + e^{-\pi t}} dt = \frac{1}{2} \sec m.$$

If we denote by x a new variable; by n a positive exponent; we substitute $e^{-\pi t} = x^n$, and, for brevity, $2mn/\pi = p$; the extreme values $t = 0$ and $t = 1/0$, will correspond to $x = 1$ and to $x = 0$, and our two equations will become

$$\int \frac{x^p - x^{-p}}{x^n - x^{-n}} \frac{dx}{x} = \frac{\pi}{2n} \tan \frac{\pi p}{2n};$$

$$\int \frac{x^p + x^{-p}}{x^n + x^{-n}} \frac{dx}{x} = \frac{\pi}{2n} \sec \frac{\pi p}{2n};$$

the integrals being taken from $x = 0$ to $x = 1$.

These formulas are due, it is known, to Euler, who proved them only for the case when the exponent p is real; the preceding analysis proves that they are still valid when that exponent is supposed imaginary.

§28. By the direct process of integration of rational functions, Euler obtained this result, noteworthy by its simplicity:

$$\int \frac{x^p + x^{-p}}{x^n + 2 \cos a + x^{-n}} \frac{dx}{x} = \frac{\pi \sin ap/n}{n \sin a \sin \pi p/n};$$

the integral being taken from $x = 0$ to $x = 1$, a being less than π , and p and n denoting positive integers such that the first is less than the second. He then observes that the same formula subsists when these exponents are arbitrary real quantities; since if x^q is substituted for x , q being a real and positive quantity, nothing is changed for the limits of integration, and defining $qn = n'$, $qp = p'$, we obtain

$$\int \frac{x^{p'} + x^{-p'}}{x^{n'} + 2 \cos a + x^{-n'}} \frac{dx}{x} = \frac{\pi \sin ap'/n'}{n' \sin a \sin \pi p'/n'};$$

and because of the indeterminate factor q , one can now take for p' and n' arbitrary real positive quantities. Euler is also lead, by induction based on the generality of analytic formulas, to view the exponent p as imaginary; replacing it by iq , he deduces this other result (Petersbourg Memoirs, years 1785 and 1787) :

$$\int \frac{\cos(q \log x)}{x^n + 2 \cos a + x^{-n}} \frac{dx}{x} = \frac{\pi}{2n \sin a} \frac{e^{\frac{aq}{n}} - e^{-\frac{aq}{n}}}{e^{\frac{\pi q}{n}} - e^{-\frac{\pi q}{n}}},$$

which he gives as a simple conjecture, observing that it would be desirable that someone would find a direct method to obtain it and verify its validity. But if we substitute, in equation (15), $e^{-t} = x^n$ and $kn = p$, the limits of integration relative to x become $x = 0$ and $x = 1$, and this equation will be identical to the preceding one, which will, in this manner, be rigorously demonstrated.

[a] S.D. Poisson, Journal de l'École Polytechnique, Cahier 18/Tome XI (1820), §28, p. 298, 308.

¹⁰ Poisson used the variables p, m, θ instead of m, p, a here, and refers also to Plana – translator.

Appendix C – Short summary by Heinrich Burkhardt [b]

The equation

$$\int_0^\infty \frac{\cos bt \, dt}{\cosh t + \cos a} = \frac{\pi}{\sin a} \frac{\sinh ab}{\sinh \pi b}$$

has already been proven by Euler [c] using a different method. He first obtains for integer values of p and n , by performing indefinite integration and some reductions, the equation¹¹

$$\int_0^\infty \frac{x^p + x^{-p}}{x^n + 2 \cos a + x^{-n}} \frac{dx}{x} = \frac{2 \pi}{n \sin a} \frac{\sin ap/n}{\sin \pi p/n}; \quad \left\{ 2 \int_0^1 \frac{x^p + x^{-p}}{x^n + 2 \cos a + x^{-n}} \frac{dx}{x} \right\}$$

then he points out that it remains unchanged when p and n are given fractional values sharing the same integer denominator¹², and he claims that it is valid for irrational and even for imaginary values of p . The first equation above is obtained after the substitution $x^n = e^t$.

[b] H. Burkhardt, *Trigonometrische Reihen und Integrale* (bis etwa 1850), *Encyclopädie der Mathematischen Wissenschaften*, Zweiter Band, Erster Teil, Zweite Hälfte, 1904-1916, p. 1133. (Quotes also Poisson and Plana)

[c] Petrop. n. a. 3 (1785[88]), p. 14 (from 1776); result without proof in 5 (1785/89) p. 14 (also from 1776). Euler extracts furthermore from this integral theorem the partial fraction decomposition of certain functions.

¹¹ Burkhardt used the variables m, r instead of p, x used here, and forgot the factor 2 on the right hand side – translator.

¹² In fact Euler proves more, reducing the case p real and n integer to the case p integer and n integer (§III); then the case p real and n real to the case p real and n integer (§XIIb), if we lend him the justification required to interchange limit and integration over $[0, 1]$, as requested in his last sentence of §III – translator.

Appendix D – List of integration formulas

Some integral formulas found in Euler's article are presented here using the Fourier notation with limits of integration, after transforming the interval $(0, 1]$ to $[0, \infty)$ with the Poisson substitution $x^n = e^{-t}$. Following Poisson and Burkhardt we also define, in order to simplify, $a = \pi - \theta$, $c = \pi - \zeta$ and $b = p/n$.

§1–§15. Main formula¹³ valid, by continuity if $a = 0$ or $b = 0$, for $|\Re a| < \pi$ and $|\Re b| < 1$:

$$S(a, b, c) := \int_0^\infty \frac{\cosh bt + \cos c}{\cosh t + \cos a} dt = \frac{1}{2} \int_{-\infty}^\infty \frac{\cosh bt + \cos c}{\cosh t + \cos a} dt = \frac{\pi \sin ab}{\sin a \sin \pi b} + \frac{a \cos c}{\sin a}.$$

§6. The proof by Euler for $-\pi < a < \pi$ and $-1 < b < 1$ uses partial fractions leading to the following elementary indefinite integration formula which can be verified by differentiation,

$$\int_0^x \frac{\sin a}{y^2 + 2y \cos a + 1} dy = A \tan \left(\frac{x \sin a}{1 + x \cos a} \right),$$

and which gives this definite integral^{14,15} by substituting $x = 1$ and $y = e^{-t}$ (first integral of §I) :

$$S(a, 0, \frac{\pi}{2}) = \int_0^\infty \frac{1}{\cosh t + \cos a} dt = \frac{a}{\sin a}, \quad |\Re a| < \pi.$$

§16. A second particular case of the main formula, $a = c = \pi/2$, is the simplest :

$$S(\frac{\pi}{2}, b, \frac{\pi}{2}) = \int_0^\infty \frac{\cosh bt}{\cosh t} dt = \frac{\pi}{2} \sec \frac{\pi b}{2}, \quad |\Re b| < 1.$$

A third particular case¹⁶, $c = \pi/2$, is studied in the rest of the article under the symbol P :

$$P(a, b) := S(a, b, \frac{\pi}{2}) = \int_0^\infty \frac{\cosh bt}{\cosh t + \cos a} dt = \frac{\pi \sin ab}{\sin a \sin \pi b}, \quad |\Re a| < \pi, |\Re b| < 1.$$

§IV. A Fourier cosine transform (of the hyperbolic secant if $a = \pi/2$) is obtained from formula P by replacing formally b with ib :

$$P(a, ib) = \int_0^\infty \frac{\cos bt}{\cosh t + \cos a} dt = \frac{\pi \sinh ab}{\sin a \sinh \pi b}, \quad |\Re a| < \pi, |\Im b| < 1.$$

§V. A two sided Laplace transform (of the hyperbolic secant if $a = \pi/2$) follows using parity :

$$\int_{-\infty}^\infty \frac{e^{\pm bt}}{\cosh t + \cos a} dt = \int_{-\infty}^\infty \frac{\cosh bt \pm \sinh bt}{\cosh t + \cos a} dt = \frac{2\pi \sin ab}{\sin a \sin \pi b}, \quad |\Re a| < \pi, |\Re b| < 1.$$

¹³ Bierens de Haan (1867) 6.20; Gradshteyn&Ryzhik (2007) 3.514.2, $0 < a < \pi$ and $0 < |b| < 1$.

¹⁴ Bierens de Haan 27.22; Gradshteyn&Ryzhik 3.514.1; Brychkov&Marichev&Prudnikov (1986) 2.3.14.29.

¹⁵ Following Weierstrass, this equation, proven twice by Euler for $-\pi < a < \pi$ (§9, §I), remains valid for $|\Re a| < \pi$ since the integral by uniform convergence and $a/\sin a$ are analytic there; their difference vanishes in any region of regularity intersecting $-\pi < a < \pi$ (Edward Charles Titchmarsh, The Theory of Functions, 2nd ed., Oxford, 1939, §4.4: Analytic continuation. Integrals containing a complex parameter, p. 147).

¹⁶ whose validity, for each fixed $|\Re a| < \pi$, can be extended again from $-1 < b < 1$ to $|\Re b| < 1$.

§VI. A first paradox concerns the periodicity in a of $\sin ab$, ignoring the condition $|\Re a| < \pi$:

$$P(a, b) = \int_0^\infty \frac{\cosh bt}{\cosh t + \cos(a \pm 2k\pi)} dt = P(a \pm 2k\pi, b) = \frac{\pi \sin(a \pm 2k\pi)b}{\sin a \sin \pi b} \quad ??$$

§VII. Limit case $a \rightarrow 0$ of formula P , which Euler verifies by reducing it to the first particular case :

$$P(0, b) = \int_0^\infty \frac{\cosh 2bt}{\cosh^2 t} dt = \pm \frac{\sinh(2b \pm 1)t}{\cosh t} \Big|_0^\infty \mp 2b P\left(\frac{\pi}{2}, 2b \pm 1\right) = \frac{\pi b}{\sin \pi b}, \quad |\Re b| < 1.$$

§VIII. The Fourier cosine transform of the hyperbolic secant squared is obtained by replacing formally b with ib in the last formula :

$$P(0, ib) = \int_0^\infty \frac{\cos 2bt}{\cosh^2 t} dt = \frac{\pi b}{\sinh \pi b}, \quad |\Im b| < 1.$$

§IX. Replacing formally a with $i \log f$ in formula P :

$$P(i \log f, b) = \int_0^\infty \frac{\cosh bt}{\cosh t + (f + 1/f)/2} dt = \frac{\pi(f^b - f^{-b})}{(f - f^{-1}) \sin \pi b}, \quad |\arg f| < \pi, |\Re b| < 1.$$

§X. Another Fourier cosine transform obtained by replacing formally b with ib in the preceding formula :

$$P(i \log f, ib) = \int_0^\infty \frac{\cos bt dt}{\cosh t + (f + 1/f)/2} = \frac{2\pi \sin b(\log f)}{(f - f^{-1}) \sinh \pi b}, \quad |\arg f| < \pi, |\Im b| < 1.$$

§XIa. Partial fraction expansion of formula P (of the secant if $\theta = \pi/2$) :

$$P(\pi - \theta, b) = \int_0^\infty \frac{\cosh bt}{\cosh t - \cos \theta} dt = \frac{\pi \sin(\pi - \theta)b}{\sin \theta \sin \pi b} = \frac{2}{\sin \theta} \sum_{k=1}^\infty \frac{k \sin k\theta}{k^2 - b^2}, \quad 0 < \theta < 2\pi, |\Re b| < 1.$$

Euler claims that he is allowed to assign purely imaginary values to b in the last series¹⁷, obtained by starting from the following recurrent series, which he had proven by the method of undetermined coefficients¹⁸ :

$$\frac{\sin \theta}{1 - 2x \cos \theta + x^2} = \sin \theta + x \sin 2\theta + x^2 \sin 3\theta + x^3 \sin 4\theta + \text{etc.}$$

§XIa. Another partial fraction expansion (of the hyperbolic secant if $\theta = \pi/2$) :

$$P(\pi - \theta, ib) = \int_0^\infty \frac{\cos bt}{\cosh t - \cos \theta} dt = \frac{\pi \sinh(\pi - \theta)b}{\sin \theta \sinh \pi b} = \frac{2}{\sin \theta} \sum_{k=1}^\infty \frac{k \sin k\theta}{k^2 + b^2}, \quad 0 < \theta < 2\pi, |\Im b| < 1.$$

§XIb. Replacing t by nt ; b by ib ; and n by i in formula P yields a second paradox, a purely imaginary value for a real integral, divergent when taken along the real axis :

$$\int_0^\infty \frac{\cosh bt}{\cos t + \cos a} dt = \frac{\pi}{i} \frac{\sinh ab}{\sin a \sinh \pi b} \quad ??$$

Poisson investigated this phenomenon¹⁹, and thus discovered examples of the Cauchy residue theorem²⁰ by changing the path of integration to avoid a singularity.

¹⁷ and, implicitly, also to the parameter b of P , a method used in its proof by Siméon Denis Poisson.

¹⁸ Leonhard Euler, Inst. Calculi Integralis IV – Eneström 660, Supplementum V, §187, p. 372.

¹⁹ S.D. Poisson, On the integrals of functions which become infinite between the limits of integration, and on the use of imaginaries in the determination of definite integrals, J. Éc. Poly. 18/XI, 1820, §33, p. 318.

²⁰ Remark by the French mathematician Claude Picard, who provided help for the French translation.